

CARNOT LIE ALGEBRAS, FIELD EXTENSIONS, AND SYSTOLIC GROWTH

YVES CORNULIER

ABSTRACT. A finite-dimensional Lie algebra over a field is Carnot if it admits a grading in the positive integers for which it is generated in degree 1. We show that, over fields of characteristic zero, this does not depend on the ground field.

As an application, we prove that if Γ is a finitely generated nilpotent group, its systolic growth is asymptotically equivalent to its growth if and only if its Malcev completion is Carnot.

1. INTRODUCTION

1.1. Carnot Lie algebras and field extensions. Fix a commutative ring R . Recall that a *Carnot-graded* Lie algebra over R is a Lie algebra \mathfrak{g} over R (in the usual sense), endowed with a grading in \mathbf{Z} , and generated by \mathfrak{g}_1 as a Lie algebra. Note that such a Lie algebra is nilpotent if and only if $\mathfrak{g}_i = 0$ for i large enough.

If \mathfrak{g} is an arbitrary Lie algebra over R , letting $(\mathfrak{g}^{(i)})_{i \geq 1}$ be its descending central series ($\mathfrak{g}^{(1)} = \mathfrak{g}$ and $\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}]$) the associated Carnot-graded Lie algebra is defined as $\bigoplus_{i \geq 1} \mathfrak{g}^{(i)} / \mathfrak{g}^{(i+1)}$ with the naturally induced bracket and grading; the Lie algebra \mathfrak{g} is *Carnot* if it is isomorphic to its associated Carnot-graded Lie algebra; in particular, for any Carnot-graded Lie algebra, the underlying Lie algebra is Carnot. For a given Carnot Lie algebra, the Carnot grading is unique up to Lie algebra automorphism but generally not unique and it is safe to distinguish between “Carnot” and “Carnot-graded”.

Carnot Lie algebras and associated Carnot-graded Lie algebras are important objects, appearing in many places under more many names (“graded”, “naturally graded”, “homogeneous”, “quasi-cyclic”), which are often inconvenient and ambiguous; the non-ambiguous “fundamental graded” is also used by some authors (with negative gradings). The use of the word “Carnot” in this context is common in sub-Riemannian and conformal geometry.

Carnot Lie algebras are ubiquitous in the study of Lie algebras and the associated Lie and discrete groups. For instance Pansu [Pan2] proved that any two quasi-isometric simply connected real Lie groups have isomorphic associated Carnot-graded real Lie algebras. The classification of various classes of nilpotent Lie finite-dimensional algebras also starts with the Carnot case: for instance

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Vergne [Ver] classified the Carnot-graded d -dimensional Lie algebras of nilpotency length exactly $d - 1$ over a field of characteristic $\neq 2$: for $d \geq 2$ there are 1 or 2 such Lie algebras, and 2 precisely when $d \geq 6$ is even; there is a large subsequent literature about refinements of this result.

If R' is a commutative R -algebra and \mathfrak{g} is a Carnot Lie algebra over R , then $R' \otimes_R \mathfrak{g}$ is Carnot as an R' -algebra. Here we are interested in the converse. Some naive counterexamples being easily available for arbitrary ground rings, let us therefore focus to Lie algebras over fields: the above observation is that being Carnot passes to field extensions; we are interested to the question whether the converse holds. Here we solve this question in the positive for finite-dimensional Lie algebras over fields of characteristic zero.

Theorem 1. *Let $K \subset K'$ be fields of characteristic zero. Let \mathfrak{g} be a finite-dimensional Lie algebra over K . Then \mathfrak{g} is Carnot over K if (and only if) $K' \otimes_K \mathfrak{g}$ is Carnot over K' .*

In other words, being Carnot does not depend on the ground field of characteristic zero. We were especially initially interested in the case of $\mathbf{R} \subset \mathbf{C}$, relevant to the classification of nilpotent Lie algebras, but we also found an application in the case of $\mathbf{Q} \subset \mathbf{R}$, to the study of lattices in nilpotent Lie groups: see the application to systolic growth provided in §1.2. There is an analogy with Sullivan's result [Sul, Theorem 12.1] that the notion of formality for a nilpotent minimal differential algebra is independent of the ground field of characteristic zero; however the proofs are not similar and we are not aware of a link between these two results.

Let us mention a few facts, in order to put the result in perspective with the classification of small-dimensional nilpotent Lie algebras (we stick to fields characteristic zero):

- Up to dimension 5, the classification is independent of the ground field, while in dimension 6 there are the smallest examples of pairs of non-isomorphic real nilpotent Lie algebras (both Carnot and non-Carnot) with isomorphic complexification. (See [Gra].)
- The smallest non-Carnot Lie algebras have dimension 5; there are 2 such isomorphism classes of Lie algebras, using the notation $(\mathfrak{l}_{5,i})$ from [Gra]:
 - The Lie algebra $\mathfrak{l}_{5,5}$ with basis $(X_i)_{1 \leq i \leq 5}$ with nonzero brackets $[X_1, X_3] = X_4$, $[X_1, X_4] = X_5$ and $[X_2, X_3] = X_5$. Its nilpotency length is 3. It is not Carnot, for instance because its center is 1-dimensional but the center of the associated Carnot-graded Lie algebra is 2-dimensional. (Note that although $\mathfrak{l}_{5,5}$ is indecomposable as a direct product, $\text{Car}(\mathfrak{l}_{5,5}) \simeq \mathfrak{l}_{5,3}$ splits as a direct product with the abelian factor generated by X_2 .)
 - The Lie algebra $\mathfrak{l}_{5,6}$ with basis $(X_i)_{1 \leq i \leq 5}$ with nonzero brackets $[X_1, X_i] = X_{i+1}$ ($i = 2, 3, 4$) and $[X_2, X_3] = X_5$. Its nilpotency length is 4. Its

associated Carnot Lie algebra $\mathfrak{l}_{5,7}$ is defined in the same way except $[X_2, X_3] = 0$.

- The smallest infinite families of non-isomorphic complex nilpotent Lie algebras occur in dimension 7; they include both families of Carnot and of non-Carnot Lie algebras.
- In dimension ≤ 6 , any complex nilpotent Lie algebra can be defined over any subfield (although not always uniquely, in dimension 6); in dimension 7 there are the smallest examples (both Carnot and non-Carnot) of complex nilpotent Lie algebras with no real form.
- In dimension ≤ 6 ; any nilpotent Lie algebra admits an invertible self-derivation (and, better, a grading in the positive integers, although not always Carnot); while in every dimension ≥ 7 there exist nilpotent Lie algebras in which every self-derivation is nilpotent (“characteristically nilpotent”).
- The full classification of nilpotent Lie algebras in dimension ≥ 8 , including the Carnot case, is open.

Over a field, every 2-nilpotent Lie algebra is Carnot (defining $\mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}]$ and \mathfrak{g}_1 any complement subspace thereof). This is not true for 3-nilpotent Lie algebra (see the example $\mathfrak{l}_{5,5}$ above). However, for 3-nilpotent Lie algebras, being Carnot can be translated into an affine condition, which allows in this case to avoid the restriction of characteristic zero in Theorem 1, see Proposition 10.

The method for proving Theorem 1 also provides the following slightly easier variant. We say that \mathfrak{g} admits a positive grading if it has a Lie algebra grading in positive integers, and that \mathfrak{g} admits a nontrivial non-negative grading if it has a Lie algebra grading in non-negative integers, not concentrated in degree 0. Important instances are Carnot gradings, but these notions are more general.

Theorem 2. *Let $K \subset K'$ be fields of characteristic zero. Let \mathfrak{g} be a finite-dimensional Lie algebra over K . Then \mathfrak{g} admits a positive grading over K if (and only if) $K' \otimes_K \mathfrak{g}$ admits a positive grading over K' . The same equivalence holds with nontrivial non-negative gradings.*

In contrast to Theorems 1 and 2, in general a \mathbf{Z} -grading cannot descend to the field of definition. For instance, \mathfrak{so}_3 admits no nontrivial grading in \mathbf{Z} defined over \mathbf{R} , while it admits one over \mathbf{C} . Here is another example with a nilpotent Lie algebra and a positive grading: let \mathfrak{g} be the 6-dimensional real Lie algebra defined as the underlying real Lie algebra of the complex 3-dimensional Lie algebra \mathfrak{h} . Then $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ is isomorphic to $\mathfrak{h} \times \mathfrak{h}$ as a complex Lie algebra. Endow the direct product $\mathfrak{l} = \mathfrak{h} \times \mathfrak{h}$ with a \mathbf{Z} -grading (valued in $\{1, 2, 3, 6\}$) for which the first factor is endowed with a Carnot grading, and the second factor is endowed with a Carnot grading multiplied by 3. Then this grading has the property that \mathfrak{l} is the direct product of $\mathfrak{l}_1 \oplus \mathfrak{l}_2$ and $\mathfrak{l}_3 \oplus \mathfrak{l}_6$. This grading does not descend to \mathfrak{g} , since otherwise \mathfrak{g} would split as a direct product of two real Heisenberg Lie algebras, which is not the case.

Every positive grading on a Lie algebra \mathfrak{g} gives rise to an invertible diagonalizable derivation, given by multiplication by j on \mathfrak{g}_j . More generally, let us call invertible \mathbf{Z} -grading of a Lie algebra \mathfrak{g} , a grading in \mathbf{Z} for which $\mathfrak{g}_0 = \{0\}$. If \mathfrak{g} is finite-dimensional over a field K of characteristic 0, a simple argument shows that \mathfrak{g} has an invertible \mathbf{Z} -grading if and only if it admits an invertible K -diagonalizable derivation. However, the analogue of Theorem 2 does not hold for invertible diagonalizable derivations.

Theorem 3. *There exists a real nilpotent Lie algebra \mathfrak{h} (of dimension 12), whose complexification admits an invertible \mathbf{Z} -grading, but not \mathfrak{h} itself.*

Remark 4. The complexification of \mathfrak{h} is an instance of a Lie algebra admitting an invertible \mathbf{Z} -grading, but no positive grading; I could not find such examples in the literature.

Remark 5. For finite-dimensional Lie algebras over an infinite field K , the existence of an invertible derivation obviously does not depend on the ground field. In characteristic zero, it is also equivalent to the existence of a semisimple invertible derivation (since one can replace a derivation by the semisimple part of its Jordan-Chevalley decomposition).

1.2. Application to the systolic growth of nilpotent groups. Let us furnish a geometric application of Theorem 1. Let Γ be a finitely generated group, and endow it with the word metric with respect to some finite generating subset S . If $\Lambda \subset \Gamma$, define its systole $\text{sys}_S(\Lambda)$ to be $\inf\{|g|_S : g \in \Lambda \setminus \{1\}\}$ (which is $+\infty$ in case $\Lambda = \{1\}$). Define, following [Gro] its *systolic growth* as the function $\sigma_{\Gamma,S}$ mapping n to the smallest index of a subgroup of systole $\geq n$ (hence $+\infty$ if there is no such subgroup). Note that Γ is residually finite if and only if $\sigma_{\Gamma,S}(n) < \infty$ for all n , and a standard argument shows that the asymptotic behavior (in the usual sense of growth of groups, see §5) of $\sigma_{\Gamma,S}$ does not depend on the choice of S ; hence we call it the systolic growth of Γ . It is obviously asymptotically bounded below by the growth (precisely, $\sigma_{\Gamma,S}(2n+1) \geq b_{\Gamma,S}(n)$, where $b_{\Gamma,S}(n)$ is the cardinal of the n -ball). It is easy to see that Γ and its finite index subgroups have asymptotically equivalent systolic growth.

It is natural to wonder when the growth and systolic growth are equivalent. Gromov [Gro, p.334] provides a simple argument, based on congruence subgroups, showing that finitely generated subgroups of $\text{GL}_d(\mathbf{Q})$ have at most exponential systolic growth. Thus for most familiar finitely generated groups of exponential growth, the systolic growth is also exponential. This leaves a lot of cases open, such as the wreath product (of exponential growth) $\mathbf{Z} \wr \mathbf{Z}$, for which an upper bound $\sigma(n) \preceq n^n$ is easily established. The discussion below now focusses on the case of polynomial growth.

We use Theorem 1 to obtain the following geometric characterization of Carnot simply connected Lie groups (among those admitting lattices).

Theorem 6. *Let G be a simply connected nilpotent real Lie group whose growth rate is polynomial of degree δ , and whose Lie algebra \mathfrak{g} definable over \mathbf{Q} . Equivalences:*

- (i) *the Lie algebra \mathfrak{g} is Carnot (over \mathbf{R})*
- (ii) *every lattice in G has systolic growth $\simeq n^\delta$*
- (iii) *some lattice in G has systolic growth $\simeq n^\delta$*
- (iv) *G admits a sequence (Γ_n) of lattices with systole $u_n \rightarrow \infty$ and covolume $\leq u_n^\delta$.*

Note that (iii) \Rightarrow (iv) is clear. The arithmeticity of lattices (see [Rag]) shows that definability over \mathbf{Q} is equivalent to the existence of a lattice, whence (ii) \Rightarrow (iii); more precisely, any lattice yields a \mathbf{Q} -structure on \mathfrak{g} . Assuming (i), we use Theorem 1 in order to show that some Carnot grading is defined over \mathbf{Q} , which allows to prove (ii). Finally the implication (iv) \Rightarrow (i) consists in rescaling G , and view the limit Ξ of the Γ_n as a lattice in the asymptotic cone of G and then observe that Γ_n is isomorphic to Ξ for n large enough. This requires some preliminaries to ensure that Ξ is indeed a lattice, and that Γ_n converges to Ξ in the space of marked groups.

The equivalence between (i) and (ii) was suggested by Gromov [Gro, p.333], with, as only comment, the easy checking of (ii) in the case of the Heisenberg group. The proof of (ii) is based on the same construction in general, but as we already mentioned, it makes, beforehand, a crucial use of Theorem 1 in its full generality, and Gromov made no hint towards proving that any of the other properties implies (i).

Using (iv), any lattice in a non-Carnot simply connected nilpotent Lie group of polynomial growth of degree δ has systolic growth $\gg n^\delta$; it would be interesting to improve this estimate. For instance, for both non-Carnot 5-dimensional Lie algebras $\mathfrak{l}_{5,5}$, $\mathfrak{l}_{5,6}$ mentioned earlier, we can check that the systolic growth is $\leq n^{\delta+1}$ (with δ the degree of growth, 8 and 11 respectively) and I do not know if it is optimal in these cases. In general, the obvious upper bound $\sigma(n) \leq n^{c \dim(G)}$, where c is the nilpotency length, given by congruence subgroups is easy to improve, but the precise behavior remains unclear and its study could shed light on how to quantify the lack of being Carnot.

2. PROOF OF THEOREMS 1 AND 2

We will focus on Theorem 1, and then describe how the proof can be adapted to yield both statements of Theorem 2.

2.1. Reduction to finite Galois extensions.

Lemma 7. *Suppose that \mathfrak{g} is a finite-dimensional Lie algebra over an infinite field K and that $\mathfrak{g} \otimes_K K(t)$ is Carnot over K , where t is transcendental. Then \mathfrak{g} is Carnot over K .*

Proof. Fix a basis $(e_1, \dots, e_k, f_1, \dots, f_\ell)$ of \mathfrak{g} , so that (f_1, \dots, f_ℓ) is a basis of $[\mathfrak{g}, \mathfrak{g}]$. Fix a Carnot grading on $(K(t) \otimes \mathfrak{g})$; then there exists a basis of $(K(t) \otimes \mathfrak{g})_1$ of the form (e'_1, \dots, e'_k) , with $e'_i = e_i + \sum_j s_{ij}(t)f_j$, where $s_{ij}(t) \in K(t)$. Let Q be the product of all the denominators of all the s_{ij} and pick $u \in K$ such that $Q(u) \neq 0$. Define $e''_i = e_i + \sum_j s_{ij}(u)f_j$ and let \mathfrak{g}_1 be the K -subspace of \mathfrak{g} with basis (e''_i) ; let \mathfrak{g}_i be the subspace generated by i -iterated brackets of \mathfrak{g}_1 . Then an easy argument shows that (\mathfrak{g}_i) is a Carnot grading of \mathfrak{g} . \square

Let us assume that the theorem is proved when $K \subset K'$ is a finite Galois extension. Then it holds for any finite extension $K \subset K'$: indeed, if $K' \subset K''$ is such that $K \subset K''$ is Galois, then $K'' \otimes_K \mathfrak{g} = K'' \otimes_{K'} (K' \otimes_K \mathfrak{g})$ is Carnot over K'' , hence by the case of finite Galois extensions, \mathfrak{g} is Carnot over K . So the result holds for all finite extensions. By Lemma 7, it also holds for all finitely generated extensions. Suppose now that $K \subset K'$ is an extension and $K' \otimes_K \mathfrak{g}$ is Carnot. Then the Carnot grading is defined over a finitely generated K -subfield K'' of K' . Hence $K'' \otimes_K \mathfrak{g}$ is Carnot over K'' , and hence, by the finitely generated case, \mathfrak{g} is Carnot over K . Thus the theorem follows from the particular case of finite Galois extensions.

2.2. Preliminaries on Levi factors and Carnot 1-parameter subgroups.

In the sequel of the proof, we use a schematic point of view: a Lie algebra is considered as defined over a ground field K , so that $\mathfrak{g}(K')$ is defined for every extension of K ; we thus avoid identifying \mathfrak{g} with the space of K -points $\mathfrak{g}(K)$.

If \mathfrak{g} is a finite-dimensional Lie algebra defined over K , the automorphism group of \mathfrak{g} defines a linear algebraic group $\text{Aut}(\mathfrak{g})$, defined over K .

If \mathfrak{g} is a finite-dimensional Lie algebra over K endowed with a grading in \mathbf{Z} defined over K , we can naturally define an action of the multiplicative group \mathbb{G}_m on \mathfrak{g} so that z acts by multiplication by z^n on \mathfrak{g}_n . Define the multiplicative character group of \mathfrak{g} as the image of \mathbb{G}_m in $\text{Aut}(\mathfrak{g})$. If \mathfrak{g} is Carnot over K , define a Carnot 1-parameter subgroup as the multiplicative character K -subgroup of $\text{Aut}(\mathfrak{g})$ associated to any Carnot grading on \mathfrak{g} .

Recall that if H is a linear algebraic group over K (of characteristic zero), it is isomorphic to a K -defined semidirect product $U \rtimes R$, with U the unipotent radical, and R reductive (possibly not connected); the factor R is unique up to $U(K)$ -conjugacy, and is called a reductive K -Levi factor [Mos]. Now let R be a K -Levi factor in $\text{Aut}(\mathfrak{g})$. Define $T = Z(R^\circ)^\circ$, where Z denotes the center and $^\circ$ denotes taking the unit connected component.

Lemma 8. *Suppose that \mathfrak{g} is Carnot over the field K of characteristic zero. Then $T = Z(R^\circ)^\circ$ contains a Carnot 1-parameter K -subgroup P of \mathfrak{g} .*

Proof. Let $P \subset \text{Aut}(\mathfrak{g})$ be a Carnot 1-parameter K -subgroup.

Let us endow \mathfrak{g} with the corresponding grading defined over K . The grading naturally induces a grading of the Lie algebra of derivations $\text{Der}(\mathfrak{g})$ in \mathbf{Z} , namely

$$\text{Der}(\mathfrak{g})_n = \{f \in \text{Der}(\mathfrak{g}) : \forall k, f(\mathfrak{g}_k) \subset \mathfrak{g}_{k+n}\}.$$

Actually this grading takes values in $\mathbf{N} = \{0, 1, \dots\}$, because \mathfrak{g} is generated by \mathfrak{g}_1 . More precisely, $\text{Der}(\mathfrak{g}) = \text{Der}(\mathfrak{g})_0 \ltimes \text{Der}(\mathfrak{g})_{\geq 1}$. The exponential of $\text{Der}(\mathfrak{g})_{\geq 1}$ consists of a unipotent closed normal subgroup. Hence the exponential of $\text{Der}(\mathfrak{g})_0$, which is equal to $\text{Aut}_{\text{grad}}(\mathfrak{g})^\circ$, contains a K -Levi factor R of $\text{Aut}(\mathfrak{g})$. Note that P is clearly contained and central in $\text{Aut}_{\text{grad}}(\mathfrak{g})$. Hence $P \subset Z(R^\circ)^\circ$. If R_1 is an arbitrary K -Levi factor, then it is conjugate to R and the corresponding conjugate of P is contained in $Z(R_1^\circ)^\circ$. \square

2.3. End of the proof of Theorem 1. Let $K \subset K'$ be a finite Galois extension of fields of characteristic zero, let \mathfrak{g} be a finite-dimensional Lie algebra defined over K and assume that \mathfrak{g} is K' -Carnot. As above, let $R \subset \text{Aut}(\mathfrak{g})$ be a K -defined Levi factor and $T = Z(R^\circ)^\circ$. By Lemma 8, there exists a 1-parameter Carnot K' -subgroup P of \mathfrak{g} contained in T (we assume $\mathfrak{g} \neq \{0\}$, so that P is K' -isomorphic to \mathbb{G}_m). The inclusion of K' -tori $P \subset T$ induces a surjective homomorphism of abelian groups $\mathbf{Z}^k \simeq \mathbf{X}(T) \twoheadrightarrow \mathbf{X}(P) \simeq \mathbf{Z}$, where $\mathbf{X}(M)$ denotes the group of multiplicative split K' -defined characters of the torus M . We extend the above map to a \mathbf{Q} -linear surjection $f : V = \mathbf{X}(T) \otimes_{\mathbf{Z}} \mathbf{Q} \rightarrow \mathbf{Q}$.

We endow \mathfrak{g} with its grading in $\mathbf{X}(T)$ induced by the action of T , which is defined over K , and which, over K' , refines the Carnot grading (in $\mathbf{X}(P)$). Denote by $\mathcal{W}(\mathfrak{g}) \subset V$ the set of weights of \mathfrak{g} , namely those $\alpha \in \mathbf{X}(T)$ such that $\mathfrak{g}_\alpha \neq \{0\}$. By faithfulness of the T -action on \mathfrak{g} , the subset $\mathcal{W}(\mathfrak{g})$ generates V \mathbf{Q} -linearly. We will use several times the following trick: if W is a subsemigroup of V , and if all the weights of \mathfrak{g}_1 (\mathfrak{g}_1 referring to the Carnot grading!) belong to W , then all the weights of \mathfrak{g} belong to W .

Let $\Gamma = \text{Gal}(K'|K)$ be the Galois group. Since R is defined over K , so is T , and hence $T(K')$ is Γ -invariant, and the action of Γ on $T(K')$ induces an action on $\mathbf{X}(T)$ and on V . Then $\mathcal{W}(\mathfrak{g})$ is a finite Γ -invariant subset of V . Considering the action of the finite group Γ on V , we write $V = V_1 \oplus V_0$, where V_1 is the set of Γ -invariant vectors and V_0 its unique invariant supplement. For $\{i, j\} = \{0, 1\}$, let p_i be the linear projection on V_i with kernel V_j . We write $f = f_0 + f_1$, where $f_i = f p_i$.

Then we have $\mathcal{W}(\mathfrak{g}_1) \subset f^{-1}(\{1\})$. Since \mathfrak{g}_1 generates \mathfrak{g} , it follows that $\mathcal{W}(\mathfrak{g}) \subset f^{-1}([1, \infty[)$. Thus for every $\gamma \in \Gamma$ and $x \in \mathcal{W}(\mathfrak{g})$, we have $f(\gamma x) \geq 1$. We deduce that

$$f_1(x) = f \left(\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma x \right) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma x) \geq 1, \quad \forall x \in \mathcal{W}(\mathfrak{g}).$$

Since $f(x) = 1$ for all $x \in \mathcal{W}(\mathfrak{g}_1)$, it follows that $f_0(x) = 1 - f_1(x) \leq 0$ for all $x \in \mathcal{W}(\mathfrak{g}_1)$. Again, since \mathfrak{g}_1 generates \mathfrak{g} , we deduce that $f_0(x) \leq 0$ for all $x \in \mathcal{W}(\mathfrak{g})$. On the other hand, we have

$$(1) \quad 0 = f_0(0) = f_0 \left(\sum_{\gamma \in \Gamma} \gamma x \right) = \sum_{\gamma \in \Gamma} f_0(\gamma x), \quad \forall x \in V.$$

If $x \in \mathcal{W}(\mathfrak{g})$, all $f_0(\gamma x)$ being ≤ 0 and their sum being zero by (1), all terms are zero; in particular for $\gamma = 1$ we obtain $f_0(x) = 0$. Since $\mathcal{W}(\mathfrak{g})$ generates V linearly, we deduce that $f_0(x) = 0$ for all $x \in V$. This proves that $f = f_1$, i.e. all components of the Cartan-grading are Γ -invariant and hence defined over K , proving the theorem.

2.4. Proof of Theorem 2. The reduction to finite Galois extensions is an immediate adaptation of the easy §2.1, and is left to the reader.

We then pursue as in §2.2 and §2.3: Lemma 8 is not necessary here; also we use the schematic language; now we define T not as in §2.3, but as a maximal K -defined torus in $\text{Aut}(\mathfrak{g})$. Then T is also a maximal torus in $\text{Aut}(\mathfrak{g})$, and enlarging K' if necessary, we can suppose that T is K' -split. By assumption, \mathfrak{g} admits a nontrivial non-negative (resp. positive) grading, which defines a 1-dimensional split torus in $\text{Aut}(\mathfrak{g})$, which we can suppose, up to conjugation, to be contained in T . The K' -split torus T provides a K' -defined Lie algebra grading of \mathfrak{g} defined over K' valued in the free abelian group of split multiplicative K' -defined characters $X(T)$, and the nontrivial non-negative (resp. positive) grading provides a homomorphism f from $X(T)$ whose image consists of non-negative integers and is not reduced to $\{0\}$ (resp. consists of positive integers). Similarly as in §2.3, we consider $f'(x) = \sum_{\gamma \in \Gamma} f(\gamma x)$. Then f' is a Γ -invariant group homomorphism from $X(T)$ to \mathbf{Z} , mapping weights to non-positive integers, and not identically zero on the set of weights (resp. mapping all weights to positive integers). This means that if we define \mathfrak{g}_n as the sum of weights spaces α for which $f(\alpha) = n$, then \mathfrak{g}_n is Γ -invariant. Hence (\mathfrak{g}_n) is a K -defined Lie algebra grading on \mathfrak{g} , which is nontrivial non-negative (resp. positive).

Remark 9. As regards the case of non-negative gradings in Theorem 2, the non-triviality statement can be strengthened with no change in the proof: if \mathfrak{i} is a characteristic ideal of \mathfrak{g} , then the existence of non-negative Lie algebra grading on \mathfrak{g} that is non-trivial on \mathfrak{i} does not depend on the ground field of characteristic zero.

3. A GENERAL CRITERION AND 3-NILPOTENT LIE ALGEBRAS

Proposition 10. *For finite-dimensional 3-nilpotent Lie algebras, Theorem 1 holds for extensions $K \subset K'$ of arbitrary fields.*

The easy proof consists in checking that being Carnot is equivalent to the existence of a solution to an affine system of equations. To achieve this goal, we present a characterization of Carnot Lie algebras amongst arbitrary nilpotent Lie algebras.

Let \mathfrak{v} be an R -module. We denote by $\mathfrak{v}^{(i)}$ the i -th component of the enveloping Lie algebra of \mathfrak{v} (if \mathfrak{v} is a free module as in the case of vector spaces over fields, this is a free Lie algebra).

If \mathfrak{g} is a Lie algebra over R and $f : \mathfrak{v} \rightarrow \mathfrak{g}$ is a R -module homomorphism, then it extends to a unique Lie algebra homomorphism from the enveloping Lie algebra; we denote its restriction to $\mathfrak{v}^{(i)}$ (valued in $\mathfrak{g}^{(i)}$) by $f^{(i)}$.

We now consider a Lie algebra \mathfrak{g} and wish to characterize when \mathfrak{g} is Carnot. A necessary condition is the existence of a supplement subspace of $[\mathfrak{g}, \mathfrak{g}]$ generating \mathfrak{g} (note that this condition holds whenever \mathfrak{g} is a nilpotent Lie algebra over a field). We now fix once and for all such a supplement subspace, which we find convenient to define as the image of an injective module homomorphism $f : \mathfrak{v} \rightarrow \mathfrak{g}$.

Denote by $\mathcal{L}(M, N)$ the module of module homomorphisms $M \rightarrow N$. If $u \in \mathcal{L}(\mathfrak{v}, \mathfrak{g}^{(2)})$, define $f_u \in \mathcal{L}(\mathfrak{v}, \mathfrak{g}^{(2)})$ by $f_u(x) = f(u) + f(x)$. It is clear that every supplement subspace of $\mathfrak{g}^{(2)}$ in \mathfrak{g} is the image of f_u for some u .

Let p_i be the projection $\mathfrak{g} \rightarrow \mathfrak{g}^{(i+1)}$. Define $\mathfrak{v}^{[i]}$ as the kernel of $p_i \circ f^{(i)}$. Note that $p_i \circ f^{(i)} = p_i \circ f_u^{(i)}$ for any $u \in \mathcal{L}(\mathfrak{v}, \mathfrak{g}^{(2)})$. Hence $\mathfrak{v}^{[i]}$ is also the kernel of $p_i \circ f_u^{(i)}$ for any $u \in \mathcal{L}(\mathfrak{v}, \mathfrak{g}^{(2)})$. Define $\Phi_i(u) \in \mathcal{L}(\mathfrak{v}^{[i]}, \mathfrak{g}^{(i+1)})$ as the restriction of $f_u^{(i)}$ to $\mathfrak{v}^{[i]}$.

Proposition 11. *Fix $k \geq 2$ and suppose that \mathfrak{g} is k -nilpotent. Then \mathfrak{g} is Carnot if and only if there exists $u \in \mathcal{L}(\mathfrak{v}, \mathfrak{g}^{(2)})$ such that $\Phi_i(u) = 0$ for all $i \in \{2, \dots, k-1\}$. Moreover, $\Phi_i(u + v) = \Phi_i(u)$ for every $i \geq 2$ and $v \in \mathcal{L}(\mathfrak{v}_1, \mathfrak{g}^{(k)})$, and Φ_{k-1} is an affine function of u .*

Proof. Suppose that \mathfrak{g} is Carnot, and fix a Carnot grading $\mathfrak{g} = \bigoplus \mathfrak{g}_i$. Then there exists u such that $\mathfrak{g}_1 = f_u(\mathfrak{v})$. Then $f_u^{(i)}$ maps $\mathfrak{v}^{(i)}$ into \mathfrak{g}_i , and hence maps $\mathfrak{v}^{[i]}$ to 0. Hence $\Phi_i(u) = 0$ for all i (note that this works without assuming nilpotency).

Conversely, suppose that there exists u with these conditions. The nilpotency of \mathfrak{g} ensures that $f_u(\mathfrak{v})$ generates \mathfrak{g} as a Lie algebra. Then the condition $\Phi_i(u) = 0$ implies that $f_u^{(i)}(\mathfrak{v}^{(i)}) \cap \mathfrak{g}^{(i+1)} = 0$ for $i \leq k-1$; note that this trivially holds for $i = 1$ and $i \geq k$. Hence the $\mathfrak{g}_i = f_u^{(i)}(\mathfrak{v}^{(i)})$ define a Carnot grading.

The condition $\Phi_i(u + v) = \Phi_i(u)$ for $i \geq 2$ is clear since v takes central values.

For the last statement, first define $\hat{\Phi}_i(u) \in \mathcal{L}(\mathfrak{v}^{[i]}, \mathfrak{g}^{(i+1)})$ to be equal to $f_u^{(i)}$, and let us check that $\hat{\Phi}_{k-1}$ is itself affine. Indeed, it is enough to show this when we evaluate on an element $[x_1, [x_2, [\dots x_{k-1}]] \dots]$. We have

$$f_u^{(k-1)}([x_1, [x_2, [\dots x_{k-1}]] \dots]) = [x_1 + u(x_1), [x_2 + u(x_2), [\dots, x_{k-1} + u(x_{k-1})]] \dots].$$

If we expand the latter term, we obtain the constant term $[x_1, [x_2, [\dots x_{k-1}]]]$, then the $k-1$ terms obtained from $[x_1, [x_2, [\dots x_{k-1}]]]$ by replacing x_j by $u(x_j)$, which depend linearly on u , and higher terms given by $(k-1)$ -iterated commutators for which at least two elements of the form $u(x_j)$ appear; since u takes values in $\mathfrak{g}^{(2)}$, these higher terms belong to $\mathfrak{g}^{(k+1)}$ and hence vanish. Therefore $\hat{\Phi}_{k-1}$ is affine, and hence Φ_{k-1} as well. \square

Proof. To check Proposition 10, just observe that if we specify Proposition 11 to 3-nilpotent Lie algebras, the condition that $\mathfrak{g} \otimes_K K'$ being Carnot over K' is

equivalent to the existence in $\mathcal{L}(\mathfrak{v}_1, \mathfrak{g}^{(2)}) \otimes_K K'$ of a solution to $\Phi_2(u) = 0$, where Φ_2 is a K -affine map, and thus boils down to the existence of a solution over K' to an affine system of equations (depending only on \mathfrak{g} and \mathfrak{v}_1 , which are fixed) with coefficients in K . From basic linear algebra, this does not depend on K' . \square

On the other hand, when \mathfrak{g} is 4-nilpotent, we see that there is a quadratic term in Φ_2 and we do not see any way to avoid the use of the Galois group (and conjugacy of tori, etc.) in the proof of Theorem 1.

Remark 12. Suppose that the ground ring is a field and \mathfrak{g} has finite dimension. If $\mathfrak{g}/\mathfrak{g}^{(i+1)}$ is free i -nilpotent, then it follows that $\mathfrak{v}^{[i]} = \{0\}$. Hence $\Phi_i(u)$ is automatically zero. In particular, if \mathfrak{g} is k -nilpotent then

- if $\mathfrak{g}/\mathfrak{g}^{(k)}$ is free $(k-1)$ -nilpotent, then \mathfrak{g} is automatically Carnot (this generalizes the fact that 2-nilpotent Lie algebras are Carnot);
- if $\mathfrak{g}/\mathfrak{g}^{(k-1)}$ is free $(k-2)$ -nilpotent, then the condition that \mathfrak{g} is Carnot reduces to the existence of u such that $\Phi_{k-1}(u)$, which is an affine condition (this generalizes the result above on 3-nilpotent Lie algebras).

Remark 13. Here when we say that being Carnot is or is not an affine condition, the Lie algebra is fixed and only the field of definition varies. Another possible point of view, not relevant our study, but natural in other contexts, is to consider the Lie algebra law as the variable.

Remark 14. As we have used above, a Carnot grading in a Lie algebra \mathfrak{g} is determined by the choice of \mathfrak{g}_1 . If 2-nilpotent and free k -nilpotent Lie algebras, \mathfrak{g}_1 can be chosen to be an arbitrary supplement subspace to $[\mathfrak{g}, \mathfrak{g}]$. However this is not the case in general, nor is it an open condition in the Grassmanian of $/\mathfrak{g}$, as the reader can check in the case of the 6-dimensional Lie algebra of upper triangular 4×4 -matrices with zero diagonal.

4. PROOF OF THEOREM 3

Here i is the complex imaginary number. We first consider, for every field K , a 12-dimensional Lie algebra \mathfrak{g}_K over K with basis

$$(x_1, \dots, x_5, x_{-1}, \dots, x_{-5}, y_1, y_{-1})$$

and nonzero brackets

$$\begin{aligned} [x_j, x_k] &= x_{j+k}, \quad ij > 0, |i| < |j|, |i+j| \leq 5; \\ [x_j, x_{-j-1}] &= y_{-1}, \quad [x_{-j}, x_{j+1}] = y_1, \quad j = 1, 2. \end{aligned}$$

Then for any three distinct basis elements x, y, z , we have $[x, [y, z]] = 0$, which immediately implies that the Jacobi identity holds. Note that the indices provide an invertible \mathbf{Z} -grading.

Proposition 15. *Suppose that K has characteristic zero. Then the Lie algebra of derivations of \mathfrak{g}_K is solvable, and every torus of automorphisms of \mathfrak{g} has dimension ≤ 1 .*

Proof. We begin by the first assertion. It is enough to show that it stabilizes a complete flag in the 4-dimensional abelianization of $\mathfrak{g} = \mathfrak{g}_K$ (which admits $(x_1, x_{-1}, x_2, x_{-2})$ as a basis). Let \mathfrak{v} be the ideal of codimension 2 containing all basis elements but $x_{\pm 1}$, and let $(\mathfrak{g}^{(i)})$ be the lower central series; note that $\mathfrak{g}^{(3)}$ has basis $(x_{\pm 4}, x_{\pm 5}, y_{\pm 1})$. Then the set of x such that $[x, \mathfrak{g}^{(3)}]$ is contained in the center of \mathfrak{g} is exactly \mathfrak{v} . Hence \mathfrak{v} is stable by every derivation. Thus the plane of the abelianization with basis (x_2, x_{-2}) is stable by every derivation of \mathfrak{g} .

Now for $x \in \mathfrak{g}$, define $C(x)$ as the set of $y \in \mathfrak{v}$ such that $[x, y] \subset \mathfrak{g}^{(3)}$; it always contains $\mathfrak{g}^{(2)}$; define $C'(x) = C(x)/\mathfrak{g}^{(2)}$ (we view $C'(x)$ as a 2-dimensional subspace of the abelianization); note that $C'(x)$ only depends on x modulo \mathfrak{v} . Then $C'(x_1) = Kx_{-2}$, $C'(x_{-1}) = Kx_2$, and $C'(\lambda x_1 + \mu x_{-1}) = \{0\}$ whenever $\lambda\mu \neq 0$. Hence the action modulo \mathfrak{v} of every automorphism of \mathfrak{g} either exchanges Kx_1 and Kx_{-1} , or preserves both. Hence the derivations of \mathfrak{g} stabilize the hyperplane with basis (x_1, x_2, x_{-2}) of the abelianization $\mathfrak{g}/\mathfrak{g}^{(2)}$. Let \mathfrak{v}' be the inverse image of the latter in \mathfrak{g} , and define $\mathfrak{w} = \{x \in \mathfrak{v} : [x, \mathfrak{v}'] \subset \mathfrak{g}^{(3)}\}$. Then $\mathfrak{g}^{(2)} \subset \mathfrak{w}$, and $\mathfrak{w}/\mathfrak{g}^{(2)}$ is the line generated by x'_2 . Hence the hyperplane with basis (x_1, x_2, x_{-2}) , the plane with basis (x_2, x_{-2}) and the line of basis (x'_2) are all invariant; this is a complete flag in $\mathfrak{g}/\mathfrak{g}^{(2)}$.

Now let us prove the second assertion. We can suppose that K is algebraically closed. Since the grading in \mathbf{Z} given by the indices provides a 1-dimensional torus of automorphism, we have to show that it is maximal. If T is a torus containing the latter, its Lie algebra is contained in the Lie algebra of degree-preserving derivations of \mathfrak{g} . Let D be such a derivation. Then D maps $y_{\pm 1}$ to a central element of the same degree, hence it induces a derivation modulo the central plane \mathfrak{z} generated by $y_{\pm 1}$. Since all homogeneous components in $\mathfrak{g}/\mathfrak{z}$ are 1-dimensional, we can write D as $x_j \mapsto j\lambda_j x_j$; here j is just a convenient normalization. Then writing the brackets shows that λ_j only depends on the sign of j , so let us write it as λ_+ and λ_- accordingly. If we turn back to \mathfrak{g} , it still holds that D maps x_j to $j\lambda_j x_j$ when $|j| \geq 2$, and maps $x_{\pm 1}$ to $\lambda_{\pm} x_{\pm 1} + a_{\pm} y_{\pm 1}$. Computing Dy_1 in two ways shows that $\lambda_+ = \lambda_-$. Hence if D_0 is the derivation defined by the grading, we see that $D = \lambda D_0 + \nu$, where ν is a nilpotent derivation (zero on all basis elements, except $x_{\pm 1} \mapsto a_{\pm} y_{\pm 1}$). Thus T is 1-dimensional. \square

We now assume that K contains an element i with $i^2 = -1$ and define the elements, for $1 \leq j \leq 5$

$$M_j = x_j + x_{-j}, \quad N_j = i(x_j - x_{-j}), \quad Y = y_1 + y_{-1}, \quad Z = i(z_1 - z_{-1}).$$

Note that we have

$$x_j = \frac{1}{2}(M_j - iN_j), \quad x_{-j} = \frac{1}{2}(M_j + iN_j), \quad y_1 = \frac{1}{2}(Y - iZ), \quad y_{-1} = \frac{1}{2}(Y + iZ).$$

In particular, $(M_1, \dots, M_5, N_1, \dots, N_5, Y, Z)$ is also a basis of \mathfrak{g}_K . Define $\varepsilon(j, k)$ to be equal to 1 if $k - j = 1$ and 0 otherwise. A computation yields that the nonzero

brackets between these new basis elements are, for $j < k$ such that $j + k \leq 5$

$$\begin{aligned} [M_j, M_k] &= M_{j+k} + \varepsilon(j, k)Y; & [M_j, N_k] &= N_{j+k} + \varepsilon(j, k)Z; \\ [N_j, M_k] &= N_{j+k} - \varepsilon(j, k)Z; & [N_j, N_k] &= -M_{j+k} + \varepsilon(j, k)Y. \end{aligned}$$

We note that this is well-defined for any field K , and we define this Lie algebra as \mathfrak{h}_K . Thus by construction, if -1 is a square in K then \mathfrak{g}_K and \mathfrak{h}_K are isomorphic.

Proposition 16. *If -1 is not a square in the field of characteristic zero K , then \mathfrak{h}_K admits no nontrivial grading in \mathbf{Z} .*

Proof. This amounts to proving that $\text{Aut}(\mathfrak{g})$ contains no K -split 1-dimensional torus. Since $\text{Aut}(\mathfrak{h})$ is solvable by Proposition 15, all its maximal tori are conjugate over K , and by the same proposition, they are all 1-dimensional. Hence it is enough to exhibit a single 1-dimensional K -anisotropic torus. Namely, we consider the derivation D mapping $M_j \mapsto N_j \mapsto -M_j$ and $Y \mapsto Z \mapsto -Y$; its eigenvalues are $\pm i$ and in particular it is not K -diagonalizable. Then the $aI + bD$ when $(a, b) \in K^2$ and $a^2 + b^2 = 1$ form a torus whose Lie algebra is the line generated by D ; this torus is K -anisotropic because D is not K -diagonalizable. \square

Therefore the Lie algebra $\mathfrak{h}_{\mathbf{R}}$ satisfies the requirements of Theorem 3.

5. GEOMETRY OF LATTICES IN NILPOTENT GROUPS

If f, g are non-negative functions defined on the integers or reals, we say that $f \preceq g$, or that f is asymptotically bounded above by g , if there exists a positive constant C such that $f(x) \leq Cg(Cx) + C$ for all x large enough. If $f \preceq g \preceq f$, we say that f, g are asymptotically equivalent and write $f \simeq g$.

We now prove Theorem 6. From the discussion in the introduction, we only have to prove (iv) \Rightarrow (i) \Rightarrow (ii).

Proof of Theorem 6, (i) \Rightarrow (ii). (Recall that two subgroups of a groups are called commensurate if their intersection has finite index in both.)

Let Γ be a lattice in \mathfrak{g} . Then Γ is commensurate to $G(\mathbf{Z})$ for some suitable algebraic embedding G into GL_d with \mathbf{Q} -defined image, which we now fix. By definition \mathfrak{g} is Carnot over the reals, which implies, by Theorem 1, that it is Carnot over the rationals. Thus fix a Carnot grading (\mathfrak{g}_i) defined over the rationals. For each i , we choose an identification between \mathfrak{g}_i and \mathbf{R}^{n_i} defined over the rationals, in such a way to ensure that the bracket $\mathfrak{g}_i \otimes \mathfrak{g}_j \rightarrow \mathfrak{g}_{i+j}$ has integer coefficients for all i, j . Given a sequence $(m_i)_{i \geq 1}$ of positive integers such that $m_i m_j$ divides m_{i+j} for all i, j , we can define, for every n coprime to m , the Lie subring $\mathfrak{h}_n = \bigoplus m_i^{-1} n^i \mathfrak{g}_i(\mathbf{Z})$. We can choose (m_i) once and for all according to the denominators in the Baker-Campbell-Hausdorff formula, to ensure that $\exp(\mathfrak{h}_n)$ is a subgroup H_n (for instance, $m_i = i!^2$ works, independently of \mathfrak{g}). The word length on G and Γ is described by Guivarch's estimates [Gui]: define on \mathfrak{g} the function $\ell(\sum_i x_i) = \sum m_i \|x_i\|^{1/i}$. Then $|\gamma| \simeq \ell(\log(\gamma))$ for $\gamma \in \Gamma$ (note that the m_i play little role here, except a convenient normalization). Clearly we have

$\min\{\ell(x) : x \in \mathfrak{h}_n\} = n$. This shows that the systole of $\exp(\mathfrak{h}_n)$, and hence of $H_n \cap \Gamma$, is asymptotically equivalent to n . On the other hand, the index of \mathfrak{h}_n in \mathfrak{h}_1 is equal to n^δ , and hence the index of H_n in H_1 is also equal to n^δ (as we see by a dévissage). It follows that the index of $H_n \cap \Gamma$ in Γ is also asymptotically equivalent to n^δ . This shows that the systolic growth of Γ is $\preceq n^\delta$. Since the growth of Γ is $\simeq n^\delta$ and the growth is an asymptotic lower bound, this concludes the proof. \square

The remainder is devoted to the proof of (iv) \Rightarrow (i).

We need some further definitions pertaining to the geometry of lattices. Let G be a group with a left-invariant distance.

Definition 17. Let B_r be the closed r -ball in G . Given a subgroup Γ of G , define its packing $\text{pack}_G(\Gamma)$ to be $\sup_{g \in G} d(g, \Gamma)$. Also define its generating radius $\text{ger}_G(\Gamma)$ as the smallest r such that Γ is generated by $\Gamma \cap B_r$.

All these definitions are understood with the usual conventions: \inf of the empty set and \sup of an unbounded subset of positive reals are $+\infty$.

Lemma 18. *Let G be a locally compact group with a continuous left-invariant geodesic distance d and let Γ be a cocompact lattice. Then $\text{ger}_G(\Gamma) \leq 2\text{pack}_G(\Gamma)$; moreover any element $\gamma \in \Gamma$ with $d(\gamma, 1) \leq n$ is a product of n elements of $B_{3\text{pack}_G(\Gamma)} \cap \Gamma$, where B_r is the closed r -ball in G .*

Proof. Fix an integer $m \geq 1$. Given $\gamma \in \Gamma$ with $d(\gamma, 1) \leq n$, consider a geodesic joining γ to 1. On this geodesic choose points $1 = x_0, \dots, x_{mn} = \gamma$ with $d(x_{i-1}, x_i) \leq \text{pack}_G(\Gamma)/m$ for all $i = 1 \dots mn$. There is γ_i in Γ with $d(x_i, \gamma_i) \leq \text{pack}_G(\Gamma)$ for all i , where we choose $\gamma_0 = 1$ and $\gamma_{mn} = \gamma$. Hence $\gamma = \prod_{i=1}^{mn} \gamma_{i-1}^{-1} \gamma_i$, and $d(1, \gamma_{i-1}^{-1} \gamma_i) \leq (2 + 1/m)\text{pack}_G(\Gamma)$ for all i . Hence $\text{ger}_G(\Gamma) \leq (2 + 1/m)\text{pack}_G(\Gamma)$; since this holds for all m we deduce $\text{ger}_G(\Gamma) \leq 2\text{pack}_G(\Gamma)$; on the other hand taking $m = 1$ in the above argument shows that γ is a product of n elements from $B_{3\text{pack}_G(\Gamma)} \cap \Gamma$. \square

Lemma 19. *Let V be a Euclidean space and Λ a lattice (in this case, $\text{ger}_V(\Lambda)$ is often denoted $\lambda_{\dim(V)}(\Lambda)$ in the literature). Then*

$$\frac{2}{\dim(V)} \text{pack}_V(\Lambda) \leq \text{ger}_V(\Lambda) \leq 2\text{pack}_V(\Lambda).$$

Proof. The right-hand inequality is borrowed from Lemma 18. For the left-hand inequality, V has a basis (e_i) with $e_i \in \Lambda$ and $\|e_i\| \leq \text{ger}_V(\Lambda)$. If $x \in V$, we write $x = \sum \alpha_i e_i$; hence we can decompose $x = w + y$ with $w \in \Lambda$ and $y = \sum \beta_i e_i$ with $|\beta_i| \leq 1/2$ for all i . Hence $\|y\| \leq \dim(V) \text{ger}_V(\Lambda)/2$, whence $\text{pack}_V(\Lambda) \leq \dim(V) \text{ger}_V(\Lambda)/2$. \square

Assume now that G is a simply connected nilpotent Lie group, endowed with a left-invariant Riemannian metric. Guivarc'h [Gui] established that the growth

rate of G and of its lattices is $\simeq n^\delta$, where δ is characterized in terms of the descending central series by $\delta = \sum_{i \geq 1} i \dim(\mathfrak{g}^{(i)})$.

The arithmeticity of lattices (see [Rag]) implies in particular that for every lattice in G , its projection on $G/[G, G]$ is also a lattice. We endow $V = G/[G, G]$ with the Euclidean metric defined by identifying $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ with the orthogonal of $[\mathfrak{g}, \mathfrak{g}]$. Let $p : G \rightarrow G/[G, G]$ be the projection, which is 1-Lipschitz. Let λ be a left Haar measure on G .

Lemma 20. *There exists a constant C (depending only on G and its Riemannian metric) such that for every lattice Γ in G and $\Lambda = p(\Gamma)$, we have $\text{pack}_G(\Gamma) \leq C \text{pack}_V(\Lambda)$.*

Proof. We argue by induction on the nilpotency length c of G . If $c = 1$ the result is trivial (with $C = 1$). Otherwise, the c -iterated commutator induces an alternating multilinear form from V^c to $G^{(c)}$, and more precisely a surjective linear map F from $V^{\otimes c}$ to $G^{(c)}$. If we endow both V and $G^{(c)}$ with their intrinsic Riemannian (Euclidean) metric, There exists a constant C_0 such that $F(v_1, \dots, v_c) \leq C_0 \prod_{i=1}^c \|v_i\|$ for all $v_1, \dots, v_c \in V$.

Note that $F(\Lambda^{\otimes c})$ is a lattice in $G^{(c)}$, of finite index in $\Gamma \cap G^{(c)}$. Moreover, it is generated by the image of the generators of $\Lambda^{\otimes c}$, and therefore is generated by elements of norm $\leq C_0 \text{ger}_V(\Lambda)^c$. Thus, using twice Lemma 19, we successively obtain $\text{pack}_{G^{(c)}}(F(\Lambda^{\otimes c})) \leq C_1 \text{ger}_V(\Lambda)^c$ for some constant $C_1 = (\dim G^{(c)})C_0/2$ and then, $\text{pack}_{G^{(c)}}(F(\Lambda^{\otimes c})) \leq C_2 \text{pack}_V(\Lambda)^c$ with $C_2 = 2^c C_1$.

On the other hand, denote by p' the projection $G \rightarrow G' = G/G^{(c)}$. If $\Gamma' = p(\Gamma)$, we have, by induction, $\text{pack}_{G'}(\Gamma') \leq C_3 \text{pack}_V(\Lambda)$ for some constant C_3 depending only on G and its fixed Riemannian metric. Thus if $x \in G$, there exists $\gamma \in \Gamma$ such that $d(p'(x), p'(\gamma)) \leq C_3 \text{pack}_V(\Lambda)$.

If we lift a minimal geodesic joining 1 to $p(x^{-1}\gamma)$, we obtain $y \in G$ such that $p(y) = p(\gamma^{-1}x)$ and $d(1, y) \leq C_3 \text{pack}_V(\Lambda)$. Since $y^{-1}\gamma^{-1}x \in G^{(c)}$, there exists $\gamma' \in F(\Lambda^{\otimes c}) \subset \Gamma$ with $d_{G^{(c)}}(\gamma'^{-1}y^{-1}\gamma^{-1}x, 1) \leq C_2 \text{pack}_V(\Lambda)^c$. Here $d_{G^{(c)}}$ is the intrinsic distance of $G^{(c)}$, which by Guivarch's estimates is distorted in such a way that $d(w, 1) \leq C_4 d_{G^{(c)}}(w, 1)^{1/c}$ for all $w \in G^{(c)}$. Hence, writing $s = \gamma'^{-1}y^{-1}\gamma^{-1}x$, we have $d(1, s) \leq C_4 C_2^{1/c} \text{pack}_V(\Lambda)$.

We have $x = \gamma y \gamma' s = \gamma \gamma' y s$, because γ' is central. Hence we have

$$d(ys, 1) \leq C \text{pack}_V(\Lambda); \quad C = C_3 + C_4 C_2^{1/c}$$

thus $d(x, \Gamma) \leq C \text{pack}_V(\Lambda)$ and accordingly $\text{pack}_G(\Gamma) \leq C \text{pack}_V(\Lambda)$. \square

Lemma 21. *For every lattice Γ in G with systole $\geq 2r + 1$, we have, denoting again $\Lambda = p(\Gamma) \subset V = G/[G, G]$, the following lower bound on its covolume*

$$\text{covol}_G(\Gamma) \geq \frac{\text{pack}_V(\Lambda) \lambda(B_r)}{2r + 1}.$$

Proof. Define a possibly finite sequence of cosets W_i of Λ by $W_1 = \Lambda$, and, assuming W_1, \dots, W_i are defined, if $d(x, \bigcup_{1 \leq j \leq i} W_j) < 2r + 1$ for all $x \in V$, then

stop; otherwise there exists, by connectedness, $x \in V$ such that $d(x, \bigcup_{1 \leq j \leq i} W_j) = 2r + 1$ and we define $W_{i+1} = x + \Lambda$.

Since the W_i are at pairwise distance $\geq 2r + 1$, the process stops, say at $i = k_r$. Since for $2 \leq i \leq k_r$ every point in W_i is at distance $2r + 1$ to a point in $\bigcup_{j < i} W_j$ and every point in V is at distance $\leq 2r + 1$ to a point in $\bigcup_i W_i$, it follows that every point in V is at distance $\leq k_r(2r + 1)$ of some point in Λ . In other words, $\text{pack}_V(\Lambda) \leq k_r(2r + 1)$.

Fix $x_i \in W_i$ and lift it to some element $g_i \in G$. Define $X = \bigcup_{1 \leq i \leq k_r} x_i B_r$. This is a disjoint union, since the x_i are at pairwise distance $\geq 2r + 1$. Moreover, the $X\gamma$ for $\gamma \in \Gamma$ are pairwise disjoint: indeed if $g_i b \gamma = g_j b' \gamma'$ with $b, b' \in B_r$ and $\gamma \neq \gamma' \in \Gamma$, then, projecting, we obtain $x_i - x_j + p(b) - p(b') = p(\gamma^{-1} \gamma') \in \Lambda$. Since $\|p(b) - p(b')\| \leq 2r$, this forces $i = j$. Thus $g_i = g_j$, hence $b\gamma = b'\gamma'$. Hence $b^{-1}b' = \gamma\gamma'^{-1} \in \Gamma$; since the systole of Γ is $\geq 2r + 1$, this implies $\gamma = \gamma'$, contradiction. This proves that the covolume of Γ is at least equal to the volume of X , and hence is $\geq k_r \lambda(B_r)$.

Combining both inequalities yields the lemma. \square

Conclusion of the proof of (iv) \Rightarrow (i). Let now (Γ_n) be a sequence of lattices in G , satisfying $\text{sys}(\Gamma_n) \geq 2u_n + 1$ and $\text{covol}(\Gamma_n) \preceq u_n^\delta$. Define $\Lambda_n = p(\Gamma_n)$ as the projection of Γ_n on $V = G/[G, G]$.

We first claim that we have $\text{pack}_V(\Lambda_n) \preceq u_n$. Indeed, we have, by Lemma 21, $\text{covol}(\Gamma_n) \geq \text{pack}_V(\Lambda_n) \lambda(B_{u_n}) / (2u_n + 1)$. Since by assumption $\text{covol}(\Gamma_n) \simeq \lambda(B_{u_n}) \simeq u_n^\delta$, we deduce that $\text{pack}_V(\Lambda_n) \preceq u_n$, proving the claim.

Lemma 20 combined with the above claim implies that $\text{pack}_G(\Gamma_n) \preceq u_n$, say $\text{pack}_G(\Gamma_n) \leq Cu_n/3$. It follows from Lemma 18 that Γ_n is generated by the elements in $B_{Cu_n} \cap \Gamma_n$, in such a way that for any integer $R \geq 1$, any element in the $B_{Ru_n} \cap \Gamma_n$ is a product of at most R elements in $B_{Cu_n} \cap \Gamma_n$.

If we divide the distance in G by u_n , the lattice Γ_n endowed with the resulting distance has the property that its systole is $\geq 2 + 1/u_n$ and that every element in the R -ball is product of at most R elements in the C -ball, and the packing of Γ_n in $(G, (1/u_n)d)$ is bounded independently of n .

By Pansu's thesis [Pan1], the $(G, (1/n)d)$ converge in the sense of Gromov-Hausdorff to a (real) Carnot simply connected nilpotent Lie group endowed with a Carnot-Carathéodory metric, with the same dimension as G (that H is isometric to a Carnot group is due to Pansu; that H inherits the group law as limit of the laws from G is proved in [Cor]). Denote by $B_H(r)$ the closed r -ball in H .

Fix any non-principal ultrafilter ω on the positive integers. The metric ultralimit Ξ of the sequence $(\Gamma_n, (1/u_n)d)$ is a discrete subset of H , with systole ≥ 2 , with the property that any element of $\Xi \cap B_H(R)$ is a product of at most R elements of $\Xi \cap B_H(C)$, for all $R \geq 1$, and any element of H is at bounded distance to some element of Ξ . The fact that Ξ is a subgroup follows from the refinement in [Cor] of Pansu's result mentioned above. Thus Ξ is a lattice in H .

Recall that a marked group on k generators is a group endowed with a map (called marking) s from $\{1, \dots, k\}$, whose image generates the group, and a net (M_i, s_i) converges to (M, s) if, denoting by N_i (resp. N) the kernel of the unique homomorphism $F_k \rightarrow M_i$ extending s_i (resp. $F_k \rightarrow M$ extending s), we have the convergence $\mathbf{1}_{N_i} \rightarrow \mathbf{1}_N$ pointwise on the set of functions $F_k \rightarrow \{0, 1\}$; see [CG] for more details.

The sequence $(\#(B(Cu_n) \cap \Gamma_n))$ being bounded, let k be an upper bound and for each n , choose a surjective map s_n from $\{1, \dots, k\}$ onto $B(Cu_n) \cap \Gamma_n$.

Since $B(Cu_n) \cap \Gamma_n$ generates Γ_n , this provides a marking of Γ_n . Define $s(i) = \lim_{n \rightarrow \omega} s_n(i) \in \Xi$ for $1 \leq i \leq k$. Then s defines a marking of Ξ : indeed every element of the R -ball in $(\Gamma_n, (1/u_n)d)$ is a product of $\leq R$ elements in the C -ball; this fact passes to the ultralimit to show that every element in the R -ball of Ξ is a product of $\leq R$ elements of S , and thus the image of s generates Ξ .

A straightforward argument (using that these groups are uniformly discrete) then shows that (Γ_n, s_n) tends to (Ξ, s) for the topology of marked groups.

Since Ξ is finitely presented, eventually Γ_n lies as a quotient of Ξ (by [CG, Lemma 2.2]), in the sense that there exists $I \in \omega$ such that for all $n \in I$, the kernel of $F_k \rightarrow \Gamma_n$ contains the kernel of $F_k \rightarrow \Xi$. Since both Ξ and Γ_n are torsion-free of the same Hirsch length, we deduce that Γ_n is isomorphic to Ξ for every $n \in I$. It follows from the rigidity of nilpotent lattices [Rag] that G is isomorphic to H , and hence that G is Carnot (i.e., \mathfrak{g} is Carnot over \mathbf{R}).

(Note that the fact that Γ_n is generated by elements of length $\preceq u_n$ —and hence bounded length after rescaling—played a crucial role: otherwise the ultralimit of the $(\Gamma_n, (1/u_n)d)$ could have been of Hirsch length less than that of G , yielding no conclusion.) \square

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CNRS – DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ PARIS-SUD, 91405 ORSAY,
FRANCE

E-mail address: `yves.cornulier@math.u-psud.fr`